T.D.C. II
Paper : IIIrd

Jacobians :
Part-2.

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Some Properties of Jacobians :-

Jacobians have the remarkable property of behaving like the derivatives of the functions of one variable. A few of important relations are given here and the proof depends upon the algebra of determinants.

We remark, that for $n=1$, the determinant is simply $\frac{\partial y_1}{\partial x_1}$ or $\frac{dy_1}{dx_1}$, i.e. also exists.

Now we will give definition of implicit function, existence theorem for implicit function (not proof) and derivative of implicit function before discussing the properties of Jacobians.

Implicit function :- Let $f(x,y)$ be a function of two variables, and $y=\phi(x)$ be a function of x so to, for every value of x, for which $\phi(x)$ is defined, $f(x,\phi(x))$ vanishes identically, i.e. $y=\phi(x)$ is a root of the functional equation $f(x,y)=0$, then $y=\phi(x)$ is an implicit function defined by the functional equation $f(x,y)=0$.

Example :- Let $f(x,y)=x^2+y^2-1$ and a point $(0,1)$ so that $f(0,1)=0$, $f_y(0,1)\neq 0$.

We have two possible solns $y=\pm\sqrt{1-x^2}$

(i) $y=+\sqrt{1-x^2}$ is the implicit function in a neighbourhood of $(0,1)$, where $|x|<1$, y

(ii) $y=-\sqrt{1-x^2}$ is the implicit function in a neighbourhood of $(0,-1)$, where $|x|<1$, $y<0$.

Existence theorem for implicit function of two variables :-

Let $f(x,y)$ be a function of two variables x and y, and (a,b) be a point of its domain of definition such that

(i) $f(a,b) = 0$

(ii) the partial derivatives f_x & f_y exists, and are continuous in a certain neighbourhood of (a,b) and

(iii) $f_y(a,b) \neq 0$

then there exists a rectangle $(a-h, a+h; b-k, b+k)$ about (a,b) such that for every value of x in the interval $[a-h, a+h]$, the equation $f(x,y) = 0$ determines one and only one value $y = \phi(x)$, lying in the interval $[b-k, b+k]$, with the following properties :

(i) $b = \phi(a)$

(ii) $f[x, \phi(x)] = 0$, for every x in $[a-h, a+h]$

(iii) $\phi(x)$ is derivable, and both $\phi(x)$ and $\phi'(x)$ i.e. derivative are continuous in $[a-h, a+h]$.

Derivative of Implicit function :-

When the equation $f(x,y) = 0$ defines y as a function of x that has a derivative $\frac{dy}{dx}$, that derivative may be obtained simply by defferentiating the equation w.r.to x, on the understanding that y is a function $y = \phi(x)$ of x. Thus we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\text{i.e. } f_x + f_y \frac{dy}{dx} = 0.$$

Jacobian of Implicit function:- If $u_1, u_2, \ldots, u_n$ and $x_1, x_2, \ldots, x_n$ are implicitly connected by n equations as:

$$\left.\begin{aligned} &f_1(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0 \\ &f_2(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0 \\ &\vdots \\ &f_n(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0 \end{aligned}\right\} \quad \text{— (i)}$$

then
$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(u_1, u_2, \ldots, u_n)} \cdot \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)} \quad \text{— (ii)}$$

Proof: Differentiating the above given relations w.r.t. $x_1, x_2, \ldots, x_n$ we obtain

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial f_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1} + \cdots + \frac{\partial f_1}{\partial u_n} \cdot \frac{\partial u_n}{\partial x_1} = 0$$

$$\left.\begin{aligned} &\Rightarrow \sum \frac{\partial f_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} = -\frac{\partial f_1}{\partial x_1} \\ &\text{and similarly } \sum \frac{\partial f_2}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} = -\frac{\partial f_2}{\partial x_2} \\ &\vdots \\ &\sum \frac{\partial f_n}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_n} = -\frac{\partial f_n}{\partial x_n} \end{aligned}\right\} \quad \text{— (iii)}$$

and so on.....

We know that by definition of Jacobians

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(u_1, u_2, \ldots, u_n)} \times \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_n} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \sum \frac{\partial f_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} & \sum \frac{\partial f_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} & \cdots & \sum \frac{\partial f_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_n} \\ \vdots & & & \\ \sum \frac{\partial f_n}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} & \sum \frac{\partial f_n}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_2} & \cdots & \sum \frac{\partial f_n}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_n} \end{vmatrix}$$

using (iii) and product of determinant.

We have

$$\frac{\partial(f_1,f_2,\cdots,f_n)}{\partial(u_1,u_2,\cdots,u_n)}\cdot\frac{\partial(u_1,u_2,\cdots,u_n)}{\partial(x_1,x_2,\cdots,x_n)}=\begin{vmatrix}-\frac{\partial f_1}{\partial x_1} & -\frac{\partial f_1}{\partial x_2} & \cdots & -\frac{\partial f_1}{\partial x_n}\\ \vdots & & & \\ -\frac{\partial f_n}{\partial x_1} & -\frac{\partial f_n}{\partial x_2} & \cdots & -\frac{\partial f_n}{\partial x_n}\end{vmatrix}$$

$$=(-1)^n\frac{\partial(f_1,f_2,\cdots,f_n)}{\partial(x_1,x_2,\cdots,x_n)}$$ By def^n of Jacobians

Which also implies that

$$\frac{\partial(u_1,u_2,\cdots,u_n)}{\partial(x_1,x_2,\cdots,x_n)}=(-1)^n\frac{\frac{\partial(f_1,f_2,\cdots,f_n)}{\partial(x_1,x_2,\cdots,x_n)}}{\frac{\partial(f_1,f_2,\cdots,f_n)}{\partial(u_1,u_2,\cdots,u_n)}}$$

Exercise: (i) If functions $u_1,u_2,\cdots,u_n$ of $x_1,x_2,\cdots,x_n$ are of the form $u_1=f_1(x_1)$; $u_2=f_2(x_1,x_2)$; $\cdots\cdots$; $u_n=f_n(x_1,x_2,\cdots,x_n)$, then prove that

$$\frac{\partial(u_1,u_2,\cdots,u_n)}{\partial(x_1,x_2,\cdots,x_n)}=\frac{\partial u_1}{\partial x_1}\cdot\frac{\partial u_2}{\partial x_2}\cdots\frac{\partial u_n}{\partial x_n}.$$

Exercise (ii): If the implicit relations are given as $f_1(x_1,x_2,\cdots,x_n,u_1)=0$; $f_2(x_2,x_3,\cdots,x_n,u_1,u_2)=0$; $\cdots\cdots$, $f_n(x_n,u_1,u_2,\cdots,u_n)=0$ then prove that

$$\frac{\partial(u_1,u_2,\cdots,u_n)}{\partial(x_1,x_2,\cdots,x_n)}=(-1)^n\frac{\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}}{\frac{\partial f_1}{\partial u_1}\cdot\frac{\partial f_2}{\partial u_2}\cdots\frac{\partial f_n}{\partial u_n}}.$$

Theorem: Let $u_1, u_2, \ldots, u_n$ be functions of n independent variables $x_1, x_2, \ldots, x_n$. The necessary and sufficient condition that the functions be connected by a relation $f(u_1, u_2, \ldots, u_n) = 0$ is that the Jacobian $\dfrac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}$ vanishes identically.

Proof: The condition is necessary: We have $f(u_1, u_2, \ldots, u_n) = 0$ — (i)

Differentiating (i) w.r.t. $x_1, x_2, \ldots, x_n$ we get

$$\left.\begin{aligned} &\frac{\partial f}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_1} + \frac{\partial f}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_1} + \cdots + \frac{\partial f}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_1} = 0 \\ &\frac{\partial f}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_2} + \frac{\partial f}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_2} + \cdots + \frac{\partial f}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_2} = 0 \\ &\vdots \\ &\frac{\partial f}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_n} + \frac{\partial f}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_n} + \cdots + \frac{\partial f}{\partial u_n}\cdot\frac{\partial u_n}{\partial x_n} = 0 \end{aligned}\right\} \text{— (ii)}$$

Eliminating $\dfrac{\partial f}{\partial u_1}, \dfrac{\partial f}{\partial u_2}, \ldots, \dfrac{\partial f}{\partial u_n}$, we obtained

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \vdots & & & \vdots \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = 0$$

$$\Rightarrow \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = 0;$$ Hence condition is necessary.

The condition is sufficient: If $J(u_1, u_2, \ldots, u_n) = 0$, then we will prove that there exists a relation between the functions $u_1, u_2, \ldots, u_n$ alone.

The equation connecting $u_1, u_2, \ldots, u_n$ and $x_1, x_2, \ldots, x_n$ are always capable, by elimination, of being transformed into the following form:

$$\left.\begin{aligned} &F_1(x_1, x_2, \ldots, x_n, u_1) = 0 \\ &F_2(x_2, x_3, \ldots, x_n, u_1, u_2) = 0 \\ &\vdots \\ &F_n(x_n, u_1, u_2, \ldots, u_n) = 0 \end{aligned}\right\} \quad \text{—— (iii)}$$

For such relations, we know that

$$J(u_1, u_2, \ldots, u_n) = (-1)^n \frac{\frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial x_2} \cdots \frac{\partial F_n}{\partial x_n}}{\frac{\partial F_1}{\partial u_1} \cdot \frac{\partial F_2}{\partial u_2} \cdots \frac{\partial F_n}{\partial u_n}} \quad \text{(By exercise (ii))} \quad \text{—— (iv)}$$

Using given condition $J(u_1, u_2, \ldots, u_n) = 0$ in eqn (iv), we have

$$\frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial x_2} \cdots \frac{\partial F_n}{\partial x_n} = 0 \quad \text{—— (v)}$$

$\Rightarrow$ We have $\frac{\partial F_r}{\partial x_r} = 0$, for some value of r; $1 \le r \le n$.

Hence for that particular value of r, the function F_r must not contain x_r. So F_r becomes

$$F_r(x_{r+1}, x_{r+2}, \ldots, x_n, u_1, u_2, \ldots, u_r) = 0 \quad \text{(using (iii))} \quad \text{—— (vi)}$$

and from (iii)

$$\left.\begin{aligned} &F_{r+1}(x_{r+1}, x_{r+2}, \ldots, x_n, u_1, u_2, \ldots, u_{r+1}) = 0 \\ &\vdots \\ &F_n(x_n, u_1, u_2, \ldots, u_n) = 0 \end{aligned}\right\} \quad \text{—— (vii)}$$

The variables $x_{r+1}, x_{r+2}, \ldots, x_n$ can be eliminated from (vi) & (vii) and consequently we have a final relation between $u_1, u_2, \ldots, u_n$ alone; which implies that there exists f s.t. $f(u_1, u_2, \ldots, u_n) = 0$. Hence the condition is sufficient.